Second-order differential equation

Find the general solution of the following second-order differential equation:

$$y'' - 4y = 2e^x + 3x$$

Solution

First, we find the homogeneous solution:

$$y'' - 4y = 0$$
$$r^2 - 4 = 0$$
$$r_1 = 2$$
$$r_2 = -2$$

The homogeneous solution is:

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

For the particular solution, we propose the following:

$$y = Ax + B + Ke^x$$

Then:

$$y' = A + Ke^x$$
$$y'' = Ke^x$$

Substituting:

$$Ke^x - 4(Ax + B + Ke^x) = 2e^x + 3x$$
$$-3Ke^x - 4Ax - 4B = 2e^x + 3x$$

Then:

$$-4A = 3$$

Which gives:

$$A = -\frac{3}{4}$$
$$B = 0$$

And:

$$-3K = 2$$
$$K = -\frac{2}{3}$$

Thus:

$$y_p = -\frac{2}{3}e^x - \frac{3}{4}x$$

The general solution is:

$$y_g = c_1 e^{2x} + c_2 e^{-2x} - \frac{2}{3}e^x - \frac{3}{4}x$$